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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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The ambiguities of the root signs can be removed only after determining the signs of x, y, z in any particular example. The signs cannot be all positive or all negative, for that would make x, y, z equal to one another. For a given value of a, there are two values of t given by the relation $t^2+t+2=a$; and for one of these values the signs before the radicals must be two positive and one negative, and for the other, one positive and two negative. For instance, when $a=2$, $t=0$, or -1, and we have the obvious solution

$$\left.\begin{aligned} x &= 2\cos\frac{2\pi}{9}, \text{ or } 2\cos\frac{2\pi}{7}; \\ y &= 2\cos\frac{4\pi}{9}, \text{ or } 2\cos\frac{4\pi}{7}; \\ z &= 2\cos\frac{8\pi}{9}, \text{ or } 2\cos\frac{8\pi}{7} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (33)$$

the sequence of signs being $+, +, -$; and $+, -, -$.

Since $\tan 3\theta = \dfrac{4t+3}{3\sqrt{3}}$, t and consequently a are rational when $3\theta = 0, \pm 30°, \pm 60°$; and in these cases the values of t and a are as under:

θ	t	a
0	$-\frac{3}{4}$	$\frac{29}{16}$
10°	0	2
− 10°	$-\frac{3}{2}$	$\frac{11}{4}$
20°	$\frac{3}{2}$	$\frac{23}{4}$
− 20°	− 3	8.

The value $\theta = 0$, gives numerical values for x, y, z. When $\theta = 10°$, a is equal to 2, and the values of x, y, z are otherwise obvious. Leaving out these two cases, we obtain the following results :—

$$\frac{-1-4\sin 10°}{2} = -\sqrt{\frac{11}{4} - \sqrt{\frac{11}{4} + \sqrt{\frac{11}{4} - \ldots}}} \quad \ldots \quad (34)$$

$$\frac{-1-4\sin 50^\circ}{2} = -\sqrt{\frac{11}{4}+\sqrt{\frac{11}{4}-\sqrt{\frac{11}{4}-\dots}}} \quad \dots \quad (35)$$

$$\frac{-1+4\sin 70^\circ}{2} = +\sqrt{\frac{11}{4}-\sqrt{\frac{11}{4}-\sqrt{\frac{11}{4}+\dots}}} \quad \dots \quad (36)$$

$$\frac{1+4\sqrt{3}\sin 20^\circ}{2} = +\sqrt{\frac{23}{4}-\sqrt{\frac{23}{4}+\sqrt{\frac{23}{4}+\dots}}} \quad \dots \quad (37)$$

$$\frac{1-4\sqrt{3}\sin 80^\circ}{2} = -\sqrt{\frac{23}{4}+\sqrt{\frac{23}{4}+\sqrt{\frac{23}{4}-\dots}}} \quad \dots \quad (38)$$

$$\frac{1+4\sqrt{3}\sin 40^\circ}{2} = +\sqrt{\frac{23}{4}+\sqrt{\frac{23}{4}-\sqrt{\frac{23}{4}+\dots}}} \quad \dots \quad (39)$$

$$-1-2\sqrt{3}\sin 20^\circ = -\sqrt{8-\sqrt{8+\sqrt{8-\dots}}} \quad \dots \quad (40)$$

$$-1-2\sqrt{3}\sin 40^\circ = -\sqrt{8+\sqrt{8-\sqrt{8-\dots}}} \quad \dots \quad (41)$$

$$-1+2\sqrt{3}\sin 80^\circ = +\sqrt{8-\sqrt{8-\sqrt{8+\dots}}} \quad \dots \quad (42)$$

6. Let us next consider the cyclic relations

$$x^2 = a + y \quad \dots \quad \dots \quad (43)$$

$$y^2 = a + z \quad \dots \quad \dots \quad (44)$$

$$z^2 = a + u \quad \dots \quad \dots \quad (45)$$

$$u^2 = a + x \quad \dots \quad \dots \quad (46)$$

and put

$$x + y + z + u = t. \quad \dots \quad \dots \quad (47)$$

Adding (43), (44), (45), (46), we have

$$s_2 = 4a + t; \quad \dots \quad \dots \quad (48)$$

multiplying (43), (44), (45), (46) by x, y, z, u respectively, and adding, we get

$$\begin{aligned} s_3 &= 4at + xy + yz + zu + ux \\ &= 4at + (x+z)(y+u); \quad \dots \quad \dots \quad (49) \end{aligned}$$

multiplying $[(43)-(45)]$ by $[(44)-(46)]$, we have

$$(x^2-z^2)(y^2-u^2)=(y-u)(z-x).$$

$$\therefore \qquad (x+z)(y+u)=-1. \qquad \dots \qquad \dots \quad (50)$$

Hence $$s_3=4at-1. \qquad \dots \qquad \dots \quad (51)$$

From the above values of s_2 and s_3, we find,

$$p_2=\tfrac{1}{2}(t^2-t-4a) \qquad \dots \qquad \dots \quad (52)$$

and $$p_3=\tfrac{1}{6}(t^3-3t^2-10at-2). \qquad \dots \quad (53)$$

If $\quad f(\xi)\equiv\xi^4-t\,\xi^3+p_2\,\xi^2-p_3\,\xi+p_4=0,$

is the equation whose roots are (x, y, z, u), then

$$f(\xi^2-a)=f(\xi)\,.\,f(-\xi) \qquad \dots \qquad \dots \quad (54)$$

Let λ, μ be the roots of $\xi^2-\xi-a=0$, as before;

then, and $$\left.\begin{aligned}\lambda+\mu&=1\\ \lambda\mu&=-a.\end{aligned}\right\} \qquad \dots \qquad \dots \quad (55)$$

We easily see by (54) that $f(-\lambda)=f(-\mu)=1$,

that is, $$\lambda^4+t\,\lambda^3+p_2\,\lambda^2+p_3\lambda+p_4=1, \qquad \dots \quad (56)$$

and $$\mu^4+t\,\mu^3+p_2\,\mu^2+p_3\,\mu+p_4=1. \qquad \dots \quad (57)$$

Subtracting (57) from (56) and dividing the result by $(\lambda-\mu)$, we obtain with the aid of (55),

$$1+2a+t(1+a)+p_2+p_3=0 \qquad \dots \quad (58)$$

on substituting in (58) the values of p_2, p_3 given by (52) and 53)

we get, $$t^3+3t+4=4at;$$

that is, $$(t+1)(t^2-t+4)=4at, \qquad \dots \qquad \dots \quad (59)$$

which shows that a is a rational function of t.

Eliminating p_3 from (56) and (57) and proceeding as in (58), we obtain

$$a(1+a)+at+ap_2+p_4=1. \qquad \dots \quad (60)$$

(52), (53), 60) give the co-efficients of $f(\xi)$ in terms of a and t. With the help of (59), we can express the co-efficients in terms of t alone, and then $f(\xi)$ takes the form,

$$\xi^4-t\,\xi^3-\frac{t^2+3t+4}{2t}\,\xi^2+\frac{t^3+2t^2+5t+8}{4}\,\xi+$$

$$1-\frac{(t^2-1)(t^2-t+4)(t^2+3t+4)}{16t^2}=0. \qquad \dots \quad (61)$$

Writing the above in the form $(a_0, a_1, a_2, a_3\ a_4)\ (\xi, 1)^4$, we find,

$$H = -\frac{(t^2+4)(3t+4)}{48t},$$

$$G = \frac{(t+2)(t^2+4)}{32},$$

$$I \equiv a_0a_4 - 4a_1a_3 + 3a_2^2$$

$$= \frac{(t^2+4)(t^2+3t+4)}{12t^2},$$

and $$3H^2 - \frac{a_0^2 I}{4} = \frac{(t^2+4)(t+2)(3t^2+2t+8)}{256t}. \quad \ldots \quad (62)$$

Hence Euler's cubic is

$$\eta^3 - \frac{(t^2+4)(3t+4)}{16t}\eta^2 + \frac{(t^2+4)(t+2)(3t^2+2t+8)}{256t} - \frac{(t+2)^2(t^2+4)^2}{4096} = 0. \quad (63)$$

Since the roots of a cyclic quartic equations are free from cubic surds, the above cubic equation must have a rational root, and this is found to be $(t^2+4)/16$, and the remaining two roots are given by the quadratic

$$\eta^2 - \frac{(t+2)(t^2+4)}{8t}\eta + \frac{(t^2+4)(t+2)^2}{256} = 0. \quad \ldots \quad (64)$$

If the roots of the cubic (63) be denoted by η_1, η_2, η_3, where

$$\left.\begin{aligned} \eta_1 &= \frac{t^2+4}{16}, \\ \eta_2 &= \frac{(t+2)(t^2+4+2\sqrt{t^2+4})}{16t}, \\ \eta_3 &= \frac{(t+2)(t^2+4-2\sqrt{t^2+4})}{16t}, \end{aligned}\right\} \quad \ldots \quad (65)$$

then the roots of the cyclic quartic (61) are found to be

$$\left.\begin{aligned} x &= \frac{t}{4} + \sqrt{\eta_1} + \sqrt{\eta_2} - \sqrt{\eta_3} \\ y &= \frac{t}{4} - \sqrt{\eta_1} + \sqrt{\eta_2} + \sqrt{\eta_3} \\ z &= \frac{t}{4} + \sqrt{\eta_1} - \sqrt{\eta_2} + \sqrt{\eta_3} \\ u &= \frac{t}{4} - \sqrt{\eta_1} - \sqrt{\eta_2} - \sqrt{\eta_3}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (66)$$

It is convenient to combine $\sqrt{\eta_2} \pm \sqrt{\eta_3}$ under the radical sign thus

$$\sqrt{\eta_2} + \sqrt{\eta_3} = \sqrt{(\eta_2 + \eta_3 + 2\sqrt{\eta_2\eta_3})}$$

$$= \sqrt{\left\{\frac{t+2}{8}\left(\frac{t^2+4}{t} + \sqrt{t^2+4}\right)\right\}}$$

and

$$\sqrt{\eta_2} - \sqrt{\eta_3} = \sqrt{(\eta_2 + \eta_3 - 2\sqrt{\eta_2\eta_3})}$$

$$= \sqrt{\left\{\frac{t+2}{8}\left(\frac{t^2+4}{t} - \sqrt{t^2+4}\right)\right\}};$$

we now obtain

$$4x = t + \sqrt{t^2+4} + \sqrt{\left\{2(t+2)\left(\frac{t^2+4}{t} - \sqrt{t^2+4}\right)\right\}}, \quad \ldots \quad (67)$$

$$4y = t - \sqrt{(t^2+4)} + \sqrt{\left\{2(t+2)\left(\frac{t^2+4}{t} + \sqrt{t^2+4}\right)\right\}}, \quad \ldots \quad (68)$$

$$4z = t + \sqrt{t^2+4} - \sqrt{\left\{2(t+2)\left(\frac{t^2+4}{t} - \sqrt{t^2+4}\right)\right\}}, \quad \ldots \quad (69)$$

$$4u = t - \sqrt{t^2+4} - \sqrt{\left\{2(t+2)\left(\frac{t^2+4}{t} + \sqrt{t^2+4}\right)\right\}}. \quad \ldots \quad (70)$$

Mr. S. Ramanujan* has given a the value 5; and then from relation (59)

$$t^3 + 3t + 4 = 4at,$$

t takes the values 4, $\sqrt{5} - 2$, and $-\sqrt{5} - 2$; for the first two of these values, x is respectively equal to

$$\tfrac{1}{2}\{2 + \sqrt{5} + \sqrt{15 - 6\sqrt{5}}\}, \quad \ldots \quad \ldots \quad (71)$$

and $\tfrac{1}{4}\{\sqrt{5}-2 + \sqrt{13-4\sqrt{5}} + \sqrt{50 + 12\sqrt{5} - 2\sqrt{65 - 20\sqrt{5}}}\}$; (72)
also the signs of x, y, z, u are $+, +, +, -$, in the first case, and $+, +, -, -$, so that (71) and (72) are the values of

$$\sqrt{5 + \sqrt{5 + \sqrt{5 - (\sqrt{5} + x)}}},$$

and

$$\sqrt{5 + \sqrt{5 - \sqrt{5 - (\sqrt{5} + x)}}}.$$

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* *Vide*, J. I. M. S., Vol. VII, page 240.

When $a = 2$, it is otherwise obvious that the three sets of values of x, y, z, u are

$$2\cos\frac{2\pi}{15},\ 2\cos\frac{4\pi}{15},\ 2\cos\frac{8\pi}{15},\ 2\cos\frac{16\pi}{15};$$

$$2\cos\frac{2\pi}{17},\ 2\cos\frac{4\pi}{17},\ 2\cos\frac{8\pi}{17},\ 2\cos\frac{16\pi}{17};$$

and
$$2\cos\frac{6\pi}{17},\ 2\cos\frac{12\pi}{17},\ 2\cos\frac{24\pi}{17},\ 2\cos\frac{48\pi}{17}.$$

The corresponding values of t derived from the relation

$$t^3 + 3t + 4 = 4\,at, \text{ are } 1,\ \frac{\sqrt{17}-1}{2}, \text{ and } \frac{-\sqrt{17}-1}{2}.$$

It is now easy to obtain.*

$$16\cos\frac{2\pi}{17} = \sqrt{17}-1+\sqrt{34-2\sqrt{17}}+2\sqrt{\{17+3\sqrt{17}-\sqrt{170+38\sqrt{17}}\}} \quad \ldots \quad (73)$$

$$16\cos\frac{4\pi}{17} = \sqrt{17}-1-\sqrt{34-2\sqrt{17}}+2\sqrt{\{17+3\sqrt{17}+\sqrt{170+38\sqrt{17}}\}} \quad \ldots \quad (74)$$

$$16\cos\frac{8\pi}{17} = \sqrt{17}-1+\sqrt{34-2\sqrt{17}}-2\sqrt{\{17+3\sqrt{17}-\sqrt{170+38\sqrt{17}}\}} \quad \ldots \quad (75)$$

$$16\cos\frac{16\pi}{17} = \sqrt{17}-1-\sqrt{34-2\sqrt{17}}-2\sqrt{\{17+3\sqrt{17}+\sqrt{170+38\sqrt{17}}\}} \quad \ldots \quad (76)$$

$$16\cos\frac{6\pi}{17} = -\sqrt{17}-1+\sqrt{34+2\sqrt{17}}+2\sqrt{\{17-3\sqrt{17}+\sqrt{170-38\sqrt{17}}\}} \quad \ldots \quad (77)$$

$$16\cos\frac{12\pi}{17} = -\sqrt{17}-1-\sqrt{34+2\sqrt{17}}+2\sqrt{\{17-3\sqrt{17}-\sqrt{170-38\sqrt{17}}\}} \quad \ldots \quad (78)$$

$$16\cos\frac{24\pi}{17} = -\sqrt{17}-1+\sqrt{34+2\sqrt{17}}-2\sqrt{\{17-3\sqrt{17}+\sqrt{170-38\sqrt{17}}\}} \quad \ldots \quad (79)$$

$$16\cos\frac{48\pi}{17} = -\sqrt{17}-1-\sqrt{34+2\sqrt{17}}-2\sqrt{\{17-3\sqrt{17}-\sqrt{170-38\sqrt{17}}\}} \quad \ldots \quad (80)$$

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* *Cf.* Hobson's *Trigonometry*, page 113; Klein's *Famous Problems in Elementary Geometry*, page 32.

7. We give below a simpler treatment of the system of equations

$$x^2 = a + y \quad \ldots \quad \ldots \quad (43)$$

$$y^2 = a + z \quad \ldots \quad \ldots \quad (44)$$

$$z^2 = a + u \quad \ldots \quad \ldots \quad (45)$$

$$u^2 = a + x. \quad \ldots \quad \ldots \quad (46)$$

From (50), we have,

$$(x + z)(y + u) = -1.$$

Put $\qquad x + z = \cot\theta,$

then, $\qquad y + u = -\tan\theta$

and $\qquad t = x + y + z + u = \cot\theta - \tan\theta$

$$= 2\cot 2\theta. \quad \ldots \quad \ldots \quad (81)$$

Adding (43) and (45), we get,

$$x^2 + z^2 = 2a + y + u.$$

$$\therefore \quad (x + z)^2 + (x - z)^2 = 4a + 2(y + u).$$

$$\therefore \quad (x - z)^2 = 4a - 2\tan\theta - \cot^2\theta. \quad \ldots \quad (82)$$

Similarly $\qquad (y - u)^2 = 4a + 2\cot\theta - \tan^2\theta. \quad \ldots \quad (83)$

(43) — (44) gives $x^2 - z^2 = y - u$;

squaring, $\qquad (x - z)^2 (x + z)^2 = (y - u)^2.$

$$\therefore \quad (x - z)^2 = \frac{(y - u)^2}{(x + z)^2} = \frac{(x - z)^2 - (y - u)^2}{1 - (x + z)^2}$$

$$= \frac{\cot^2\theta - \tan^2\theta + 2(\cot\theta + \tan\theta)}{\cot^2\theta - 1},$$

from (82) and (83),

$$= \sec^2\theta\,(1 + \tan 2\theta). \quad \ldots \quad (84)$$

Similarly, $\qquad (y - u)^2 = \operatorname{cosec}^2\theta\,(1 + \tan 2\theta). \quad \ldots \quad (85)$

Since $\qquad x + z = \cot\theta$, and $y + u = -\tan\theta$,

we get from (84) and (85).

$$2x = \cot\theta + \sec\theta\sqrt{1 + \tan 2\theta}, \quad \ldots \quad (86)$$

$$2y = -\tan\theta + \operatorname{cosec}\theta\sqrt{1 + \tan 2\theta}, \quad \ldots \quad (87)$$

$$2z = \cot\theta - \sec\theta\sqrt{1 + \tan 2\theta}, \quad \ldots \quad (88)$$

$$2u = -\tan\theta - \operatorname{cosec}\theta\sqrt{1 + \tan 2\theta}, \quad \ldots \quad (89)$$

where $\qquad \theta = \tfrac{1}{2}\cot^{-1}\dfrac{t}{2},$

and from (82) and (84),

$$4a = 2\tan\theta + \cot^2\theta + \sec^2\theta\,(1 + \tan 2\theta). \quad (90)$$

$$= -2\cot\theta + \tan^2\theta + \operatorname{cosec}^2\theta\,(1+\tan 2\theta). \quad (91)$$

$$= t^3 + 3 + 4t^{-1},$$

after some transformations.

The change from x to y, y to z, etc. is equivalent to changing θ into $\left(\theta - \frac{\pi}{2}\right)$. With the aid of (90) and (91), it is easy to verify that the above values of x, y, z, u do actually satisfy the relations

$$x^2 = a + y,\ y^2 = a + z, \text{ etc.};$$

the values obtained in (67), (68), (69) and (70) are not equally of verification.

8. Let us now consider the cyclic quintic

$$f(\xi) \equiv \xi^5 - t\,\xi^4 + p_2\,\xi^3 - p_3\,\xi^2 + p_4\,\xi - p_5 = 0, \quad \dots \quad (92)$$

whose roots are x, y, z, u and v satisfying the relations

$$\left.\begin{aligned} x^2 &= a + y \\ y^2 &= a + z \\ z^2 &= a + u \\ u^2 &= a + v \\ v^2 &= a + x. \end{aligned}\right\} \quad \dots \quad \dots \quad (93)$$

Adding these we have

$$s_2 = 5a + t. \quad \dots \quad \dots \quad (94)$$

Multiplying them by x, y, z, u, v, and adding we get

$$s_3 = at + l, \quad \dots \quad \dots \quad \dots \quad (95)$$

where l stands for the cyclic expression

$$xy + yz + zu + uv + vx.$$

Squaring each and adding,

$$\begin{aligned} s_4 &= s_2 + 2at + 5a^2 \\ &= 5a^2 + 5a + t\,(2a + 1). \end{aligned} \quad \dots \quad (96)$$

Again squaring each, multiplying by x, y, z, u, v, and adding, we obtain

$$s_5 = a^2t + 2al + y^2x + z^2y + u^2z + v^2u + x^2v.$$

Multiplying (93) by v, x, y, z, u and adding,

we have, $$y^2x + z^2y + u^2z + v^2n + x^2v = at + m,$$

where m denotes the cyclic expression

$$xz + yu + zv + ux + vy.$$

$$\therefore \qquad s_5 = a^2t + at + 2al + m. \quad \ldots \quad \ldots \quad (97)$$

Evidently, $l + m = p_2$.

From the values of $s_2, s_3, s_4, s_5,$ we find,

$$2p_2 = t^2 - t - 5a \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (98)$$

$$6p_3 = t^3 - 3t^2 - 13at + 2l \quad \ldots \quad \ldots \quad \ldots \quad (99)$$

$$24p_4 = t^4 - 6t^3 - t^2(22a-3) + t(18a-6) + 15a(3a-2) + 8lt. \quad (100)$$

$$120p_5 = t^5 - 10t^4 - 15t^3(2a-1) + t^2(70a-18) + t(149a^2 - 126a - 12)$$
$$- 60a + l(20t^2 - 20t - 52a - 24). \quad \ldots \quad (101)$$

We have now expressed the co-efficients of the cyclic quintic (92) in terms of a, t and l. From the relation,

$$f(\xi^2 - a) = -f(\xi).f(-\xi),$$

we find two linear relations among the co-efficients of the quintic by proceeding as in § 3 and § 6. These are

$$(1 + 3a + a^2) + t(1 + 2a) + p_2(1 + a) + p_3 + p_4 = 0, \quad \ldots \quad (102)$$

and $$(a + 2a^2) + t(a + a^2) + ap_2 + ap_3 + p_5 = 1. \quad \ldots \quad (103)$$

Substituting in (102), (103) the values of $p_2, p_3, p_4, p_5,$ we get

$$8l(1+t) + t^4 - 2t^3 + t^2(3-10a) + t(2a+6) + 9a^2 - 18a + 24 = 0, \quad (104)$$

and $$4l(5t^2 - 5t - 3a - 6) + t^5 - 10t^4 + t^3(15 - 10a) + t^2(70a - 18)$$
$$+ t(9a^2 - 66a - 12) - 60(a^2 - a + 2) = 0. \quad \ldots \quad (105)$$

Eliminating l from (104) and (105), we obtain the following relation between a and t :—

$$9a^3 - a^2(19t^2 + 17t + 40) + a(11t^4 + 18t^3 + 19t^2 + 24t + 28)$$
$$= t^6 + t^5 + 3t^4 + 11t^3 + 44t^2 + 33t + 32. \quad \ldots \quad (106)$$

The calculations are tedious, and it does not appear that the result (106) can be obtained by a shorter process. When a is zero, the

expression on the right-hand side of (106) vanishes, as shown otherwise by Mr. M. T. Naraniengar in the discussion of the cyclotomic equation,*

$$\frac{x^{31}-1}{x-1}=0.$$

As a further verification, take $a = 2$; the six sets of values of x, y, z, u, v are easily seen to be,

$$\left.\begin{array}{l}
2\cos\frac{2\pi}{31},\ 2\cos\frac{4\pi}{31},\ 2\cos\frac{8\pi}{31},\ 2\cos\frac{16\pi}{31},\ 2\cos\frac{32\pi}{31};\\
2\cos\frac{6\pi}{31},\ 2\cos\frac{12\pi}{31},\ 2\cos\frac{24\pi}{31},\ 2\cos\frac{48\pi}{31},\ 2\cos\frac{34\pi}{31};\\
2\cos\frac{10\pi}{31},\ 2\cos\frac{20\pi}{31},\ 2\cos\frac{40\pi}{31},\ 2\cos\frac{18\pi}{31},\ 2\cos\frac{36\pi}{31};\\
2\cos\frac{2\pi}{33},\ 2\cos\frac{4\pi}{33},\ 2\cos\frac{8\pi}{33},\ 2\cos\frac{16\pi}{33},\ 2\cos\frac{32\pi}{33};\\
2\cos\frac{5\pi}{33},\ 2\cos\frac{10\pi}{33},\ 2\cos\frac{20\pi}{33},\ 2\cos\frac{40\pi}{33},\ 2\cos\frac{14\pi}{33};\\
2\cos\frac{2\pi}{11},\ 2\cos\frac{4\pi}{11},\ 2\cos\frac{8\pi}{11},\ 2\cos\frac{16\pi}{11},\ 2\cos\frac{10\pi}{11}.
\end{array}\right\} \quad (107)$$

The result of (106) now reduces to

$$(t^3+t^2-10t-8)\,(t^2-t-8)\,(t+1)\dagger=0.$$

The first factor corresponds to the first three sets of values in (107), the second factor to the next two sets, and the third factor to the last set.

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* *A special class of Equations*; J. I. M. S., Vol., VII, page 84, Result III.

† *Ibid*, Result II.

'On Borel's Method for Double Series'*

By V. THIRUVENKATACHARYA, M.A., L.T.

Part I

1. This paper is a continuation of my article on Summable Double Series contributed to the Fourth Mathematical Conference, the immediate object here being to extend Borel's method to Double Series (Part I) and to express the sums of certain double series in terms of the Bernoullian and Eulerean numbers (Part II). For example, it is proved in the course of the paper that

$$\overset{\infty}{\underset{n=0}{S}}\ \overset{\infty}{\underset{m=0}{S}} \left[(-)^{m+n} (m+n)^{2s+1} \right] = (-)^{s} \frac{2^{2s+2}-1}{2s+2} B_{s+1}$$

where B_{s+1} is the $(s+1)$th Bernoullian number,

2. *Borel's definition of the sum of a Summable Double Series*:—

Let us write
$$\begin{aligned} S_{m,n} = {} & U_{0,0} + U_{0,1} + U_{0,2} + \dots + U_{0,n} \\ & + U_{1,0} + U_{1,1} + U_{1,2} + \dots + U_{1,n} \\ & + \quad\ + \qquad\qquad\qquad\qquad\ + \\ & + U_{m,0} + \qquad\qquad\qquad\quad + U_{m,n} \end{aligned}$$

and let us consider the expression

$$\frac{\partial^2}{\partial x\,\partial y}\left[e^{-x-y}\sum_0^\infty\sum_0^\infty S_{m,n}\frac{x^m\,y^n}{m!\,n!}\right]\dagger$$

$$= e^{-x-y}\sum_0^\infty\sum_0^\infty \left\{ S_{m+1,\,n+1} - S_{m+1,\,n} - S_{m,\,n+1} + S_{m,\,n} \right\} \frac{x^m\ y^n}{m!\ n!}$$

$$= e^{-x-y}\sum_0^\infty\sum_0^\infty U_{m+1,\,n+1}\frac{x^m\ y^n}{m!\ n!} = e^{-x-y}\frac{\partial^2}{\partial x\,\partial y}\,U(x,y). \quad \dots \quad (2.1)$$

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* Contributed to the Science Congress held at Bombay, 1926.

† Either $$\frac{\partial}{\partial x}\left[e^{-x-y}\sum_0^\infty\sum_0^\infty S_{m,n}\frac{x^m y^n}{m!\ n!}\right] = e^{-x-y}\frac{\partial}{\partial x}\,U(x,y)$$

or $$\frac{\partial}{\partial y}\left[e^{-x-y}\sum_0^\infty\sum_0^\infty S_{m,n}\frac{x^m y^n}{m!\ n!}\right] = e^{-x-y}\frac{\partial}{\partial y}\,U(x,y)$$

will lead to the same result.

Integrating (2.1) with respect to x and y between the limits 0 to x and 0 to y respectively, we have

$$e^{-x-y} \sum_0^\infty \sum_0^\infty S_{m,n} \frac{x^m\, y^n}{m!\, h!} - e^{-x} \sum_0^\infty S_{m,0}\, \frac{x^m}{m!} - e^{-y} \sum_0^\infty S_{0,n} \frac{y^n}{n!} + S_{0,0}$$

$$= \int_0^y \int_0^x e^{-\cdot -y} \frac{\partial^2}{\partial x\, \partial y} U(x, y)\, dx\, dy. \qquad \dots \quad (2.2)$$

But it has been shown already* that $\int_0^\infty \int_0^\infty e^{-x-y} \frac{\partial^2}{\partial x\, \partial y} U(x, y)\, dx\, dy$

is associated with $\qquad \underset{1}{\overset{\infty}{S}}\ \underset{1}{\overset{\infty}{S}}\ U_{m,n}.$

Therefore, we see that if $\underset{1}{\overset{\infty}{S}}\ \underset{1}{\overset{\infty}{S}}\ U_{m,n}$ is summable, then the double limit

$$\lim_{\substack{x \to \infty \\ y \to \infty}} \left[e^{-x-y} \sum_0^\infty \sum_0^\infty S_{m,n} \frac{x^m y^n}{m!\ n!} - e^{-x} \sum_0^\infty S_{m,0} \frac{x^m}{m!} \right.$$

$$\left. - e^{-y} \sum_0^\infty S_{0,n} \frac{y^n}{n!} + S_{0,0} \right] \quad (2.3)$$

exists and is equal to $\underset{1}{\overset{\infty}{S}}\ \underset{1}{\overset{\infty}{S}}\ U_{m,n}$.

Now (2.3) is equal to

$$\lim_{\substack{x \to \infty \\ y \to \infty}} e^{-x-y} \sum_0^\infty \sum_0^\infty S_{m,n} \frac{x^m y^n}{m!\ n!} - \lim_{x \to \infty} e^{-x} \sum_0^\infty S_{m,0} \frac{x^m}{m!}$$

$$- \lim_{y \to \infty} e^{-y} \sum_0^\infty S_{0,n} \frac{y_n}{n!} + U_{0,0}.$$

From the properties of Single Summable Series,† it follows that

$$\lim_{x \to \infty} e^{-x} \sum_0^\infty S_{m,0} \left[\frac{x^m}{m!} \right] = \underset{m=0}{\overset{m=\infty}{S}} {}_{m,0}$$

and

$$\lim_{y \to \infty} e^{-y} \sum_0^\infty S_{0,n} \left[\frac{y^n}{n!} \right] = \underset{n=0}{\overset{n=\infty}{S}} {}_{0,n}.$$

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* *Vide Summable Double Series,* "Journal of the Indian Mathematical Society," Vol. XV. p. 248.

† *Vide* Bromwich, page 268, article 99.

It has been shown already* that if $\overset{m=\infty}{\underset{m=1}{S}}\ \overset{n=\infty}{\underset{n=0}{S}}\ U_{m,n}$ and $\overset{\infty}{\underset{0}{S}}\ U_{m,0}$ are both summable, then $\overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}}\ U_{m,n}$ is summable. It is quite easy to show in a similar manner that if $\overset{\infty}{\underset{1}{S}}\ \overset{\infty}{\underset{1}{S}}\ U_{m,n}$ and $\overset{\infty}{\underset{0}{S}}\ U_{m,0}$ are both summable, $\overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{1}{S}}\ U_{m,n}$ is summable. If in addition to this, $\overset{\infty}{\underset{0}{S}}\ U_{0,n}$ is summable, then $\overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}}$ is also summable

$$\therefore \lim_{\substack{x\to\infty\\ y\to\infty}} e^{-x-y} \sum_0^\infty \sum_0^\infty S_{m,n} \frac{x^m y^n}{m!\,n!}$$

$$= \overset{\infty}{\underset{1}{S}}\ \overset{\infty}{\underset{1}{S}}\ U_{m,n} + \overset{\infty}{\underset{1}{S}}\ U_{0,n} + \overset{\infty}{\underset{1}{S}}\ U_{m,0} - U_{0,0}$$

$$= \overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}}\ U_{m,n}. \qquad \ldots \qquad \ldots \quad (2.4)$$

We have now arrived at the analogue of Borel's for a double series and established that the sum $\overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}}\ U_{m,n}$ is given by the first member of (2.4).

Let us take for example :—

$$\begin{aligned} S = 1 - t + t^2 - t^3 + &\ \ldots\ldots \\ - t + t^2 - t^3 + t^4 &\ \ldots\ldots \\ + t^2 - t^3 + t^4 - t^5 &\ \ldots\ldots \\ \ldots\ldots\ldots\ldots &\ \ldots\ldots \end{aligned} \qquad \ldots \quad (2.5)$$

Then $\quad S_{m,n} = \dfrac{1 + (-)^m t^{m+1} + (-)^n t^{n+1} + (-)^{m+n} t^{m+n+2}}{(1+t)^2}$

$$\therefore \sum_{m=0}^\infty \sum_{n=0}^\infty S_{m,n} \frac{x^m y^n}{m!\,n!} = \frac{1}{(1+t)^2} \left\{ e^{x+y} + t\,e^{y-xt} + te^{x-yt} + t^2 e^{-xt-yt} \right\} \qquad (2.6)$$

$$\therefore \lim_{\substack{x\to\infty\\ y\to\infty}} e^{-x-y} \left[\sum_0^\infty \sum_0^\infty S_{m,n} \frac{x^m y^n}{m!\,n!} \right] = \frac{1}{(1+t)^2}, \text{ if } t > -1.$$

$$\therefore \qquad S = \frac{1}{(1+t)^2}, \text{ if } t > -1. \qquad \ldots \quad (2.7)$$

As special cases, let $t=1$ and 2 respectively in (2.5): then we have

$$\begin{aligned} \tfrac{1}{4} = 1 - 1 + 1 - 1 + &\ \ldots\ldots \\ - 1 + 1 - 1 + 1 &\ \ldots\ldots \\ + 1 - 1 + 1 - 1 &\ \ldots\ldots \\ \ldots\ldots\ldots\ldots &\ \ldots\ldots \end{aligned}$$

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* *J. I. M. S.*, Vol. XV, page 252.

and

$$\begin{aligned} \tfrac{1}{9} = \; & 1 - 2 + 2^2 - 2^3 + \quad \cdots \\ & - 2 + 2^2 - 2^3 + 2^4 - \quad \cdots \\ & + 2^2 - 2^3 + 2^4 - 2^5 + \quad \cdots \\ & \cdots\cdots, \cdots\cdots \end{aligned}$$

agreeing with the results obtained in *J. I. M. S*, Vol. XV, page 229.

PART II.

3. *Uniform Summability, Differentiation and Integration of Summable Double Series*:—Let us suppose that the terms of a Summable Double Series are functions of two variables θ and ϕ (or only of one variable θ, which is only a special case of the previous one), so that we may write

$$\sum_0^\infty \sum_0^\infty U_{m,n}(\theta, \phi) \frac{x^m y^n}{m!\, n!} = U(x, y; \theta, \phi).$$

We can now say that $\underset{m=0}{\overset{\infty}{S}} \underset{n=0}{\overset{\infty}{S}} U_{m,n}(\theta, \phi)$ is uniformly summable with respect to θ and ϕ in the region $\alpha < \theta < \beta$ and $\gamma < \phi < \delta$ provided that the integral $\int_0^\infty \int_0^\infty e^{-x-y} U(x, y; \theta, \phi)\, dx\, dy$ converges uniformly in the region.

As a special case we can extend Weirstrass's **M** test to the double integral and state that the double series is uniformly summable if we can find a positive function $M(x, y)$ independent of θ and ϕ such that

$$U(x, y; \theta, \phi) < M(x, y), \text{ if } \alpha < \theta < \beta \text{ and } \gamma < \phi < \delta,$$

and

$$\int_0^\infty \int_0^\infty e^{-x-y} M(x, y)\, dx\, dy$$

is convergent and satisfies de la Vallee Poussin's conditions.

For, in that case, we can choose p and q independent of θ and ϕ so that

$$\int_p^\infty \int_q^\infty e^{-x-y} M(x, y)\, dx\, dy < \varepsilon.$$

$$\therefore \quad \left| \int_p^\infty \int_q^\infty e^{-x-y} U(x, y; \theta, \phi)\, dx\, dy \right|$$

$$< \int_p^\infty \int_q^\infty e^{-x-y} M(x, y)\, dx\, dy < \varepsilon$$

$$\therefore \quad \int_0^\infty \int_0^\infty e^{-x-y} U(x, y; \theta, \phi)\, dx\, dy$$

converge uniformly, and therefore $\overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}}\ U_{m,n}(\theta, \phi)$ is uniformly summable in the region $\alpha < \theta < \beta$ and $\gamma < \phi < \delta$.

THEOREM (A): *If all the terms* $U_{m,n}(\theta, \phi)$ *are continuous and the double series*

$$\overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}}\ U^{m,n}(\theta, \phi)$$

is uniformly summable and

$$\sum_0^x \sum_0^x U_{m,n}(\theta, \phi) \frac{x^m y!}{m!\ n!}$$

satisfies Pringshiem's conditions and is uniformly convergent for all finite values of x *and* y *for* $\alpha < \theta < \beta$ *and* $\gamma < \phi < \delta$, *the equation*

$$\int_\alpha^\beta \int_\gamma^\delta \overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}}\ U_{m,n}(\theta, \phi)\, d\theta\, d\phi = \overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}} \int_\alpha^\beta \int_\gamma^\delta U_{m,n}(\theta, \phi)\, d\theta\, d\phi$$

will be true. ... ... (3.1)

For

$$\int_\alpha^\beta \int_\gamma^\delta \overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}}\ U_{m,n}(\theta, \phi)\, d\theta\, d\phi = \int_\alpha^\beta \int_\gamma^\delta \int_0^\infty \int_0^\infty e^{-x-y}\, U(x, y;\ \theta, \phi)\ dx\, dy\, d\theta\, d\phi$$

$$= \int_0^\infty \int_0^\infty e^{-x-y}\, dx\, dy \int_\alpha^\beta \int_\gamma^\delta U(x, y;\ \theta, \phi)\, d\theta\, d\phi^*$$

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* Since $\int_0^\infty \int_0^\infty e^{-x-y}\, U(x, y;\ \theta, \phi)\ dx\ dy$ is convergent uniformly and satisfies de la Vallee Poussin's conditions, M and N can be chosen so that for all values of $p > M$ and for all values of $q > N$

$$\left| \int_p^\infty \int_q^\infty e^{-x-y}\, U(x, y;\ \theta, \phi)\ dx\ dy \right| < \varepsilon.$$

$$\therefore \left| \int_\alpha^\beta \int_\gamma^\delta \int_p^\infty \int_q^\infty e^{-x-y}\, U(x, y;\ \theta, \phi)\ dx\ dy\ d\theta\ d\phi \right| < \varepsilon\, (\beta - \alpha)\, (\delta - \gamma).$$

Now $\int_\alpha^\beta \int_\gamma^\delta \int_0^p \int_0^q e^{-x-y}\, U(x, y;\ \theta, \phi)\ dx\ dy\ d\theta\ d\phi$

$$= \int_0^p \int_0^q \int_\alpha^\beta \int_\gamma^\delta e^{-x-y}\, U(x, y;\ \theta, \phi)\ d\theta\ d\phi\ dx\ dy$$

as the limits of integration are finite constants. Now make p and q tend to ∞. Then

$$\int_\alpha^\beta \int_\gamma^\delta \int_0^\infty \int_0^\infty - \underset{\substack{p\to\infty \\ q\to\infty}}{\mathrm{Lt}} \int_\alpha^\beta \int_\gamma^\delta \int_p^\infty \int_q^\infty = \int_0^\infty \int_0^\infty \int_\alpha^\beta \int_\gamma^\delta$$

As the limit tends to zero, the required result is obtained.

on account of the uniform convergency of the integral.

Now $\int_\alpha^\beta \int_\gamma^\delta U(x, y; \theta\phi)\, d\theta\, d\phi = \int_\alpha^\beta \int_\gamma^\delta \sum_0^\infty \sum_0^\infty U_{m,n}(\theta, \phi) \frac{x^m y^n}{m!\, n!}\, d\theta\, d\phi$

$$= \sum_0^\infty \sum_0^\infty \int_\alpha^\beta \int_\gamma^\delta U_{m,n}(\theta, \phi) \frac{x^m y^n}{m!\, n!}\, d\theta\, d\phi\,*$$

since $U(x, y;\ \theta, \phi)$ is uniformly convergent for all finite values of x and y and satisfies Pringshiem's conditions.

$\therefore$ The first member of (3.1)

$$= \int_0^\infty \int_0^\infty e^{-x-y}\, dx\, dy \sum_0^\infty \sum_0^\infty \int_\alpha^\beta \int_\gamma^\delta U_{m,n}(\theta, \phi) \frac{x^m y^n}{m!\, n!}\, d\theta\, d\phi$$

$$= \mathop{S}_0^\infty \mathop{S}_0^\infty \int_\alpha^\beta \int_\gamma^\delta U_{m,n}(\theta, \phi)\, d\theta\, d\phi. \quad \ldots \quad \ldots \quad (3.2)$$

---

* Since $U(x, y;\ \theta, \phi)$ is uniformly convergent for all finite values of x and y and satisfies Pringshiem's conditions, we have

$$\left| \sum_p^\infty \sum_q^\infty U_{m,n}(\theta, \phi) \frac{x^m y^u}{m!\, n!} < \varepsilon \right|$$

irrespective of the order of summation

$$\therefore \quad \left| \sum_p^\infty \sum_q^\infty \int_\alpha^\beta \int_\gamma^\delta U_{m,n}(\theta, \phi) \frac{x^m y^n}{m!\, n!}\, d\theta\, d\phi \right| < \varepsilon(\beta-\alpha)(\delta-\gamma).$$

We have also that corresponding to a given ε, p aad q may be chosen that

$$\left| U(x, y;\ \theta, \phi) - \sum_0^{p-1} \sum_0^{q-1} U_{m,n}(\theta, \phi) \frac{x^m y^n}{m!\, n!} \right| < \varepsilon.$$

$$\therefore \quad \left| \int_\alpha^\beta \int_\gamma^\delta U(x, y;\ \theta, \phi)\, d\theta\, d\phi - \sum_0^{p-1} \sum_0^{q-1} \int_\alpha^\beta \int_\gamma^\delta U_{m,n}(\theta, \phi) \frac{x^m y^n}{m!\, n!}\, d\theta\, d\phi \right| < \varepsilon(\beta-\alpha)(\delta-\gamma)$$

$$\therefore \quad \left| \int_\alpha^\beta \int_\gamma^\delta U(x, y;\ \theta, \phi)\, d\theta\, d\phi - \sum_0^\infty \sum_0^\infty \int_\alpha^\beta \int_\gamma^\delta U_{m,n}(\theta, \phi) \frac{x^m y^n}{m!\, n!}\, d\theta\, d\phi \right| < 2\varepsilon(\beta-\alpha)(\delta-\gamma).$$

THEOREM (B): *If the same conditions as in Theorem* (A) apply to $\overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}}\ \frac{\partial}{\partial\theta}\, U_{mn}(\theta, \phi)$ then the equation

$$\frac{\partial}{\partial\theta}\ \overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}}\ U_{mn}(\theta, \phi) = \overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}}\ \frac{\partial}{\partial\theta} U_{mn}(\theta, \phi)$$

will be true.

The proof follows as in Theorem (A).

4. Let us consider the double series $\overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}}\ x^m y^n$, where x and y are complex numbers of absolute value not less than unity. It follows easily that

$$\overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}}\ x^m y^n = \int_0^\infty \int_0^\infty e^{-t-p}\, e^{xt} y^{yt}\, dt\, dp = \frac{1}{(1-x)(1-y)},$$

and hence it is inferred that the integral for $\overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}}\ x^m y^n$ will converge uniformly and the double series is a uniformly summable series and is continuous in x and y so long as R (x) and R (y) are each less than unity.

If we put $x = e^{i\theta}$ and $y = e^{i\phi}$, we find that

$$\frac{\partial^{r+s}}{\partial\theta^r\, \partial\phi^s}\, x^m y^n = i^{r+s}\, m^r . n^s . x^m y^n.$$

Thus the integral function associated with $\overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}}\ \frac{\partial^{r+s}}{\partial\theta^r\,\partial\theta^s}\, x^n y^n$ is

$$U_{r,s}(xt, yp) = i^{r+s} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} m^r n^s \frac{(xt)^m (yp^n)}{m!\ n!}.$$

Now $m^r = m(m-1) \ldots (m-r+1) + m(m-1) A_1$

$$\ldots (m-r+2) + \ldots + m A_{r-1}$$

and $n^s = n(n-1) \ldots (n-s+1) + n(n-1) B_1$

$$\ldots (n-s+2) + \ldots \quad n B_{s-1}.$$

$$\therefore \quad i^{r+s} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{m^r n^s (xt)^m (yp)^n}{m!\ n!}$$

$$= i^{r+s}\, e^{t}\, e^{yp} \left[(xt)^r + A_1 (xt)^{r-1} \ldots \ldots + A_{r-1}\, xt\right]$$

$$\times \left[(yp)^s + B_1 (yp)^{s-1} + \ldots \ldots B_{s-1}\, yp\right]. \quad (4.1)$$

$$\therefore \quad |U_{r,s}(xt, yp)| \leqslant e^{t\cos\theta}\, e^{p\cos\phi} \left[t^r + |A_1|\, t^{r-1} + \ldots + |A_{r-1}|\, t\right]$$

$$\times \left[p^s + |B_1|\, p^{s-1} + \ldots + |B_{s-1}|\, p\right]$$

$$\therefore \int_0^\infty \int_0^\infty e^{-t-p}\, U_{r,s}(xt, yp)\, dt\, dp$$

$$\leqslant \int_0^\infty \int_0^\infty e^{-t(1-\cos\theta)} e^{-p(1-\cos\phi)}$$

$$[t^r + \ldots + |A_{r-1}|\, t]\ [p^s + \ldots + |B_{s-1}|\, p]\, dp\, dt$$

and this converges uniformly it $\cos\theta \leqslant (1-a)$ and $\cos\phi \leqslant (1-b)$ where a and b are positive quantities however small.

5. Let us apply the results of the previous articles to some of the double series of which the sums have been already found.

(i) Now $\quad \overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}}\ (-)^{m+n} \sin(m+n)\theta = \frac{1}{2}\tan\frac{\theta}{2}.$

It has been shown in article 4, that the double series* on the left-hand side is uniformly summable and the derivatives of the same are also uniformly summable, and therefore it can be differentiated any number of times with respect to θ.

$$\overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}}\ (-)^{m+n} \sin(m+n)\theta = \tfrac{1}{2}\tan\frac{\theta}{2}.$$

$$= \frac{2^2-1}{2!}B_1\theta + \frac{2^4-1}{4!}B_2\theta^3 + \frac{2^6-1}{6!}B_3\theta^5 + \ldots\ \ldots, \quad (5.1)$$

if θ is not an odd multiple of π.

Differentiating this $2s-1$ times and putting $\theta = 0$, we have

$$\left.\begin{matrix} 0 - 1^{2s-1} + 2^{2s-1} - 3^{2s-1} + \ \ . \ \ . \ \ . \\ - 1^{2s-1} + 2^{2s-1} - 3^{2s-1} + 4^{2s-1} - \ \ . \ \ . \ \ . \\ + 2^{2s-1} - 3^{2s-1} + 4^{2s-1} - 5^{2s-1} + \ \ . \ \ . \ \ . \\ - \ \ . \ \ . \ \ . \ \ . \ \ . \ \ . \ \ . \ \ . \ \ . \ \ . \ \ . \end{matrix}\right\} = (-)^{s-1}\frac{2^{2s}-1}{2s}. \quad (5.2)$$

Again $\quad \dfrac{\sec\theta}{4} = \cos\theta - \cos 3\theta + \cos 5\theta \ \ . \ \ . \ \ . \ \ . \ \ . \ \ . \quad (5.3)$

$$\begin{matrix} - \cos 3\theta + \cos 5\theta - \cos 7\theta + \ \ . \ \ . \ \ . \ \ . \\ + \cos 5\theta - \cos 7\theta + \cos 9\theta \ \ . \ \ . \ \ . \ \ . \ \ . \\ - \qquad\qquad + \qquad\qquad - \ \ . \ \ . \ \ . \ \ . \ \ . \end{matrix}$$

* This is a special case of $\overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}}\ x^m y^n$ when $x = y$.

Expanding $\sec\theta$ as a power series in θ and differentiating both sides of (5.3) and putting $\theta = 0$, we have

$$\begin{aligned} \tfrac{1}{4}(-)^s E_s = {} & 1^{2s} - 3^{2s} + 5^{2s} \quad . \; . \; . \; . \\ & - 3^{2s} + 5^{2s} - 7^{2s} \quad . \; . \; . \; . \\ & + 5^{2s} - 7^{2s} + 9^{2s} \quad . \; . \; . \; . \\ & - \qquad + \qquad - \quad . \; . \; . \; . \end{aligned} \tag{5.4}$$

This is an extension of Cesaro's result to Double Series.

(ii) From the identity

$$\tfrac{1}{2}\{\cos(m\theta + n\phi) + \cos(m\theta + n\phi)\} = \cos m\theta \cos n\phi,$$

we can get easily that

$$\overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}} \cos m\theta \cos n\phi = \tfrac{1}{4}.$$

This double series as well as its derivatives with respect to θ and ϕ are uniformly summable. We can therefore differentiate the double series with respect to θ and ϕ any number of times.

$$\therefore \qquad \overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}} m^{2s} n^{2r} \cos m\theta = 0 \qquad \ldots \qquad \ldots \tag{5.5}$$

and

$$\overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}} m^{2s+1} n^{2r+1} \sin m\theta \sin n\phi = 0, \qquad \ldots \tag{5,6}$$

provided that θ and ϕ are not multiples of 2π.

Put $\theta = \pi$ and $\phi = \pi$ in (5.5) and $\theta = \phi = \dfrac{\pi}{2}$ in (5.6). Then

$$\overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}} (-)^{m+n} m^{2s} n^{2r} = 0,$$

and

$$\overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}} (-)^{m+n} (2m+1)^{2s+1} (2n+1)^{2r+1} = 0. \ \ldots \tag{5.7}$$

6. As an instance of integration, take the series

$$\overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}} \sin(2m + 2n + 1)\theta = \frac{\operatorname{cosec}\theta}{4}, \qquad \ldots \qquad \ldots \tag{6.1}$$

where $0 < \theta < \pi$.

Integrate this between the limits θ to $\dfrac{\pi}{2}$,

$$\overset{\infty}{\underset{0}{S}}\ \overset{\infty}{\underset{0}{S}} \frac{\cos(2m + 2n + 1)\theta}{2m + 2n + 1} = \tfrac{1}{4} \log \cot \frac{\theta}{2}, \qquad \ldots \tag{6.2}$$

if $0 < \theta < \pi$.

The Algebraic (2,2) Correspondence.

By Dr. R. Vaidyanathaswamy.

[This paper is mainly a presentation in connected form of the leading properties of the (2,2) correspondence to be found in a pamphlet printed and circulated privately by the author about six years ago. In the light of the copious literature on this subject, the special features of this work appear to be :—

(1) the method of deriving the algebraic properties of the correspondence from its normal form,

(2) the parametric treatment of Poncelet's Theorem,

and (3) the explicit use of the operational calculus for the (2,2) correspondence.

Historically one may distinguish three main aspects in which the subject has been approached. The first relates to the addition-formula of elliptic functions, the second to the Poncelet porism of polygons which are inscribed in one conic and circumscribed to another. These two aspects merge into one another in the work of Jacobi, Cayley, Clifford, Halphen, and several later writers; on the other hand, there have also been noteworthy contributions of a purely algebraic character to the theory of Poncelet polygons. The third aspect is the more modern one of Invariant-theory.

The following memoirs might be profitably consulted in reference to the topics of this paper :—

(*a*) *Algebraic view-point.*

(1) M. Maurice Fouché : *Sur la transformation doublement quadratique, les polygones de Poncelet, et l'involution multiple.* Bull. Soc. Math de la France, Vol. 44, 1916.

(2) K. Rohn. *Das Schliessungsproblem von Poncelet und eine gewisse Erweiterung* : Bericht. Ver. K. S. Ges. der Wiss. zu Leipzig, 60-61, 1908—1909, page 94.

In the latter a recurrence formula for the closure conditions for two conics is obtained, in a more useful form than in Halphen,

Traite des Fonctions Elliptiques Tome II, and several properties of the closure invariants are derived by ingenious geometrical reasoning. While finding the order of the closure condi ion, the author however fails to find an expression for the order of the *reduced closure condition*, namely, the condition that a conic may be inscr'bed to a *proper r-gon* inscribed in a given conic; this latter order is found in the present paper as 'the number of distinct special rth roots of the identical transformation.' The only reference to this order that I know of, in the literature, occurs in a Presidential Address of Prof. H. S. White, where he mentions the correct result, without indicating any references*

There is also an elementary treatment of Poncelet polygons by means of the (2,2) correspondence in Darboux, *Principes des Geometrie Analytique.*

(*b*) For the *Invariant-geometric view-point*, the following should be consulted:

(1) E. Kasner: *Invariant-theory of the Inversion Group.* Trans. Am. Math. Soc, Vol. 1, 1900.

(2) H. W. Turnbull: *Geometrical Interpretation of the complete system of the* (2,2) *Form*: Proc. Roy. Soc. of Edin., 1924.

(3) R. Vaidyanathaswamy: *On the Rank of the Double-binary Form.* Proc. Lond. Math. Soc. Ser. 2, Vol. 21,

(4) H. W. Turnbull: *Double-binary Forms IV*: Proc. Edin Math. Soc. 1923-4, Part II.

The normal form which is fundamental in this paper is shewn in ( ) to hold for double-binary forms of higher order as well. The paper (1) of E. Kasner is of great importance as it presents the several aspects of the subject in mutual relation to one another.

The 'product' and 'sum' of two correspondences, and the rules for the related calculus will be found in Severi-Löffler, *Verlesungen über algebraische Geometrie*, page 170. The product and sum are represented for the purposes of the present paper by the symbols $\otimes$ and $\times$, instead of as usual, by $\times$ and $+$. By this change of notation, we keep in touch with the algebraic definition. Thus if two correspondences C_1, C_2 between two binary domains (x) and (y) are defined by

$$C_1(x, y) = 0$$
$$C_2(x, y) = 0,$$

* (*Science*, Feb. 4, 1916, pp. 149—158).

then the *sum* of the correspondences, which is symbolised in our notation by $C_1 \times C_2$, is also defined algebraically by the equation

$$C_1(x, y) \times C_2(x, y) = 0.$$

The algebraic process corresponding to the symbol $\otimes$ is an elimination.

Reduction of the double-binary Quadratic

1. The general double-binary quadratic may be written

$$F(xy) = L\,x_1^2 + 2M\,x_1x_2 + N\,x_2^2$$

where L, M, N are binary quadratics in y_1y_2.

The variables x, y occurring here are to be supposed quite independent. We have first to inquire whether there exist linear transformations of x, y respectively which transform $F(xy)$ into a multiple of itself.

Consider a binary form $(a_0x_1^n + a_1x_1^{n-1}.x_2 + \ldots + a^nx_2^n)$ and suppose x_1x_2 to be subjected to a linear transformation δ. The set of quantities $(x_1^n, x_1^{n-1}x_2, \ldots, x_2^n)$ will thereby be subjected to a linear transformation δ' and the set $(a_0\,a_1 \ldots a_n)$ which is contragredient to the set $(x_1^n, x_1^{n-1}x_2 \ldots, x_2^n)$ will be subjected to a transformation δ'' which is the conjugate of the inverse of δ'. It is obvious of course that δ'' is not the most general transformation of the a's. The condition that given transformation of the a's may be a transformation δ'', that is may be derivable from a linear transformation of x_1x_2, is given by the following Theorem ;—

*Any linear transformation of the co-efficients of a binary form which keeps the discriminant of the form invariant is equivalent to a linear transformation of the variables.**

The easiest proof of this theorem is by a geometrical representation. Let the form $a_0x_1^n + a_1x_1^{n-2}x_2 + \ldots$ be represented by the point whose homogenous co-ordinates are $(a_0a_1 \ldots a_n)$ in S_n a space of n dimensions. The forms which are perfect nth powers will then correspond to points on a rational norm curve Γ in S_n. The discriminant of the form will

* This theorem will occupy an important place in the theory (due to Hurwitz, *Math Ann*. Bd. 45, p. 381) of the transformations on the co-efficients of a form, which are 'induced' by linear transformations on the variables ; the concept of such 'induced' tranformations is fundamental in the invariant-theory of forms ; see, for instance, Weitzenbuck ***Invarianten-theorie***, p, 8,

correspond to the locus Δ generated by the osculating sub-planes or $(n-2)$ flats of Γ. We notice that Γ is related to Δ in an intrinsic geometric way—somewhat in the same way as the cuspidal edge is related to a developable in three dimensions. Hence any transformation of Δ into itself must transform Γ into itself. Thus any linear transformation of the a's which keeps the discriminant invariant, must transform perfect powers into perfect powers and is therefore equivalent to a linear transformation of the variables.

To apply this theorem to the double quadratic F (xy): denote its y-discriminant and x-discriminant by δ_x, δ_y respectively, so that $\delta_y = LN - M^2$; and denote the apolar quadratic factors of the sextic co-variants of δ_x, δ_y by $(P_x\, Q_x\, R_x)$, $(P'_y\, Q'_y\, R'_y)$ respectively. There are three invalutoric transformations I'_1, I'_2, I'_3 of y which transform δ_y into a multiple of itself; namely, these are the transformations

$$\text{(i)} \qquad y'_1 = \frac{dP'_y}{dy_2}$$

$$y'_2 = -\frac{dP'_y}{dy_1}$$

and two others formed similarly from Q'_y, R'_y. These transformations $I'_1 I'_2 I'_3$ of y give rise therefore to linear transformations of L, M, N which keep the discriminant δ_y invariant; and hence (by the theorem) are equivalent to certain linear transformations $I_1 I_2 I_3$ of x. Moreover these are the only linear transformations of y which have this property. Reversing this argument, we see that $I_1 I_2 I_3$ must be the involutoric transformations derived in the same way from $P_x Q_x R_x$ in some order say, in the order $P_x Q_x R_x$, as $I'_1 I'_2 I'_3$ from $P'_y Q'_y R'_y$. If $\equiv$ denote equality except for a constant factor, we see then

$$F(x, I'_r y) \equiv F(I_r x, y)$$

or, in other words $F(x, y) \equiv F(I_r x, I'_r y)$ $(r = 1, 2, 3)$.

Thus F admits a dihedral group (composed of identity and three operations of order 2) of simultaneous transformations of x and y. On this account, $(I_1 I_2 I_3 I'_1 I'_2 I'_3)$ will be termed '*the group involutions*' and (PQR P′Q′R′) '*the group quadratics*' of F. Also quartics belonging to the standard pencils* of δ_x, δ_y will be called *x- and y- standard* quartics.

* If f be any quartic and H its Hessian, the pencil $\lambda f + \mu H$ may be conveniently termed 'a standard pencil.' In other words, a standard pencil of quartics would be a pencil which contains *three* perfect squares.

The discriminant quartics δ_x, δ_y will themselves be called '*branch quartics*' of F.

The theorem $F(xy) \equiv F(I_r x, I_r' y)$, is evidently equivalent to the statement:

The x-eliminant of $F(xy)$ and any standard x-quartic is the product of two standard y-quartics; and vice versa.

In other words, F, regarded as a transformation in accordance with the equation $F(xy) = 0$, not only transforms each x into two y's but also each standard x-quartette into two standard y-quartettes.

As a particular case, the eliminant of $F(xy)$ and δ_x is the square of a standard y-quartic Since P_x^2 is a standard x-quartic, we have as another particular case: The eliminant of $F(xy)$ and P_x is a standard y-quartic. By using the theorem $F(xy) \equiv F(I_1 x, I'_1 y)$, we can see that the eliminant of $F(xy)$ and a linear factor of P_x is a quadratic apolar to the associated group quadratic P'_y.

2. Writing the equation $F(xy) \equiv F(I_r x, I_r' y)$ as an identity

$$F(x, y) = K_r . F(I_r x, I'_r y)$$

it is easy to see that K_r is a function of the determinants of I_r and I'_r only. Hence if $F(x, y)$ and $\phi(x, y)$ admit the same pairs of group-involutions, so does $[\lambda F(x, y) + \mu \phi(x, y)]$. In other words, *the family of double-quadratic forms which have given pairs of group-quadratics (P, P'), (Q, Q'), (R, R') is a linear family.*

To determine the number of parameters in this family, suppose $F(x, y)$ is known to admit the group-involutions I_1, I'_1. Transform (x, y) to new variables (X, Y) in such wise that I_1, I'_1 take the form:

$$I_1 : \begin{pmatrix} X_1 = X'_2 \\ X_2 = X'_1 \end{pmatrix} ; I'_1 : \begin{pmatrix} Y_1 = Y'_2 \\ Y_2 = Y'_1 \end{pmatrix} .$$

Let $F(xy)$ become $F'(XY)$. Then the equation

$$F'(XY) \equiv F'(I_1 X, I'_1 Y)$$

shows that the matrix of F' has to be symmetric about the non-leading diagonal: this gives three linear equations between the co-efficients of F. Similarly we obtain three more linear equations from the involutions I_2, I'_2. The third set I_3, I'_3 gives no new equations, since it is merely the operational product of the first two sets.

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NOTES AND QUESTIONS.

Notes and Questions.

On the Signs of Terms in a Determinant.

Prof. Purushotham's Note entitled 'A new Rule for the Expansion of a Determinant,' which appeared in the last number of this Journal, raises the question as to what really, in practice, is the most economical method of reckoning the sign of any assigned term in a determinant $| a_{nn} |$. The only principle which appears to have been utilised for this purpose is one which is nearly as old as the Theory of Determinants, namely, that the sign of any term $a_{1p_1} a_{2p_2} \cdots a_{np_n}$ is *plus* or *minus*, according as the number of inversions in the permutation $(p_1\, p_2 \cdots p_n)$ is even or odd. For an elaborate discussion of this principle, and the consequent rules for determining the sign, reference may be made to the chapters on 'the *affect* of sequences' in the first volume of Cullis: *Matrices and Determinoids.*

It is well-known that the number of transpositions (or simple interchanges of two symbols) necessary to pass from the fundamental permutation $(12 \ldots n)$ to $(p_1\, p_2 \cdots p_n)$ is either *always* odd or *always* even; the permutation $(p_1\, p_2 \cdots p_n)$ is said to be of *odd* or *even* class in the two cases. Now, it is clear that this number of transpositions is indeterminate (except for remaining always odd or always even), and has no superior limit; it is equally clear that it must have an inferior limit. It can be shewn that *the least number of transpositions by means of which we can pass from* $(12 \ldots n)$ *to* $(p_1\, p_2 \cdots p_n)$ *is equal to the number of inversions in the latter.** We can conclude from this, that of all rules for the signs of terms, in which the procedure is the counting of individual transpositions, the shortest is the usual rule which counts the number of inversions—which is also the rule mentioned in Prof. Purushottam's note. But though it is the shortest, it can hardly be called short, as it involves $\frac{1}{2}n(n-1)$ examinations or separate mental acts. A better

* For proof, see for instance Cullis: l. c.

Rule can be stated which does not proceed by counting transpositions.

THEOREM I. A cyclic permutation on r letters is of even or odd class according as $r-1$ is even or odd.

Proof: The number of inversions in the permutation $(23\ldots r1)$ of $(123\ldots r)$ is directly seen to be $r-1$.

THEOREM II. If the permutation $(p_1\, p_2 \ldots p_n)$ of $(12 \ldots n)$ can be decomposed into k cyclic permutations,* then it is of even or odd class, according as $(n-k)$ is even or odd.

Proof: Let the k cyclic components involve respectively $r_1, r_2, \ldots r_k$ symbols, so that $r_1 + r_2 + \ldots + r_k = n$. It is clear that the r_λ letters involved in any of these cyclic permutations can be restored within the permutation $(p_1\, p_2 \cdots p_n)$ to their natural order by means of $r_\lambda - 1$ transpositions (Th. I.). Thus, we can pass from $(p_1\, p_2 \cdots p_n)$ to $(12 \ldots n)$ by $\Sigma\,(r_\lambda - 1) = (n-k)$ transpositions, which proves the theorem.

The rule to be enunciated is an immediate consequence of this theorem. We may state the rule in two forms.

Rule I. Write down any term of the determinant $|\,a_{nn}\,|$. Begin by striking out any one of the n elements in the term: say, the element a_{pq}. Next strike out that element in the term whose first suffix is q: suppose it is a_{qr}. Next strike out the element whose first suffix is r. Proceeding in this way, one is bound sooner or later to return to the element a_{pq} which was struck out first. As soon as this happens, count *one*. Then begin by striking out any of the remaining elements, and proceed similarly till an element already struck out is reached again then count *two*; and so on. If the number thus counted is k when all the elements have been struck out, the sign of the term is $(-1)^{n-k}$.

We may notice that any diagonal element (of the type a_{pp}) which may be present in the term, contributes nothing to the sign of the term.

* Such a decomposition into cyclic permutations is unique. See, for instance, Burnside: *Theory of Groups*, page 2. The Theorem is true only if every symbol which remains in its original place, is also counted as a cyclic permutation (on *one* symbol).

Accordingly such elements may be struck out initially, and the process carried out for the remaining elements, provided the value of n is reduced by the number of elements so struck out.

RULE II. Mark the elements of any assigned term in the determinant itself. Begin by ticking off any row; then take the marked element in this row and proceed upwards or downwards along its column till a diagonal element is reached, and tick the row of this diagonal element. Similarly take the column of the marked element in this row, and tick the row of the diagonal element which it contains. Proceed in this wise till a row already ticked is reached; then count *one*. Begin again by ticking any of the rows which remain unticked, and proceed similarly till a row already ticked is reached, then count *two*, and so on, till all the rows have been ticked. Let the final result of the counting be k; the sign of the term is then $(-1)^{n-k}$.

The only counting involved in this rule is that of the number k, which is necessarily less than n, and usually a small number. It is clear therefore that the net labour involved would be considerably less than in the counting of inversions.

R. VAIDYANATHASAWMY.

Some Results involving Prime Numbers.

1. Results involving primes only, are comparatively small in number. In the *Quarterly Journal of Mathematics* (Vols. 27 and 28) Glaisher has elaborated results obtained by Rogel (*Ed. Times Reprint*, Vol. LV, p. 66). Prof. Hallberg in the first of a series of papers (J. I. M. S., Vol. IX) has given a very elegant result involving primes only.

2. The following results may be found interesting:—

(i) Consider the identity

$$\frac{\pi^2}{6} = S_2 \equiv \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \dots$$

$$= \frac{1}{\left(1 - \frac{1}{2^2}\right)} \cdot \frac{1}{\left(1 - \frac{1}{3^2}\right)} \cdot \frac{1}{\left(1 - \frac{1}{5^2}\right)} \dots \dots$$

where 2, 3, 5 ... ... are the prime numbers

This is obviously rewritten as under :—

$$S_2 = \frac{1}{1^2} + \frac{1}{2^2} S_2 + \frac{1}{3^2}\left(1 - \frac{1}{2^2}\right) S_2 + \frac{1}{5^2}\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) S_2$$

$$+ \frac{1}{7^2}\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right) S_2 + \dots \dots$$

since on transposition and simplification this reduces to

$$S_2\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right) \dots = 1.$$

Hence we have

$$\frac{1}{2^2} + \frac{2^2 - 1}{2^2 . 3^2} + \frac{(2^2 - 1)(3^2 - 1)}{2^2 . 3^2 . 5^2} + \frac{(2^2 - 1)(3^2 - 1)(5^2 - 1)}{2^2 . 3^2 . 5^2 . 7^2} \dots$$

$$= \frac{S_2 - 1}{S_2} = 1 - \frac{6}{\pi^2} \qquad (1)$$

(ii) Generally

$$S_{2n} = \sum_{r=1}^{\infty} \frac{1}{r^{2n}} = \frac{2^{2n-1} B_n \pi^{2n}}{(2n)!}$$

where B_n is the nth Bernoullian number and the general result corresponding to (1) is

$$\frac{1}{2^{2n}}+\frac{2^{2n}-1}{2^{2n}.3^{2n}}+\frac{(2^{2n}-1)(3^{2n}-1)}{2^{2n}.3^{2n}.5^{2n}}+\frac{(2^{2n}-1)(3^{2n}-1)(5^{2n}-1)}{2^{2n}.3^{2n}.5^{2n}.7^{2n}}+\ldots$$

$$=1-\frac{1}{S_n}=1-\frac{(2n)!}{2^{2n-1}B_n\pi^{2n}}\quad\ldots\qquad\ldots\ (2)$$

in which primes only occur.

(iii) The last result, may also be written:

$$\frac{1}{3^{2n}}+\frac{3^{2n}-1}{3^{2n}.5^{2n}}+\frac{(3^{2n}-1)(5^{2n}-1)}{3^{2n}.5^{2n}.7^{2n}}+\frac{(3^{2n}-1)(5^{2n}-1)(7^{2n}-1)}{3^{2n}.5^{2n}.7^{2n}.11^{2n}}$$

$$+\ldots=1-\frac{2\,(2n)!}{(2^{2n}-1)B_n\pi^{2n}}\ \ldots\quad(3)$$

where 3, 5, 7, 11, 13, 17, 19, etc. are all the odd primes.

These series are obviously convergent.

3. (i) Next take the result:

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\ \ldots$$

$$=\frac{1}{\left(1+\frac{1}{3}\right)}\,\frac{1}{\left(1-\frac{1}{5}\right)}\,\frac{1}{\left(1+\frac{1}{7}\right)}\,\frac{1}{\left(1+\frac{1}{11}\right)}\cdots\quad\cdots$$

where 3, 5, 7 ... are the odd primes.

Transforming as before, we get

$$\frac{4}{\pi}-1=\frac{1}{3}-\frac{1}{5}\left(1+\frac{1}{3}\right)+\frac{1}{7}\left(1+\frac{1}{3}\right)\left(1-\frac{1}{5}\right)$$

$$+\frac{1}{11}\left(1+\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)--+\ldots$$

i.e. $$\frac{4}{\pi}-1=\frac{1}{3}-\frac{3+1}{3.5}+\frac{(3+1)(5-1)}{3.5.7}+\frac{(3+1)(5-1)(7+1)}{3.5.7.11}$$

$$-\frac{(3+1)(5-1)(7+1)(11+1)}{3.5.7.11.13}++-+\ldots\ \ldots\quad(4)$$

Here it will be noticed that the factors of the numerator in each term are of the form $p+1$ or $p-1$ according as p is a prime of the form $4k+3$ or $4k+1$, while the sign of each term as a whole is $+$ or $-$ acoording as the last term in the denominator is a prime of the form $4k+3$ or $4k+1$.

(ii) In general,

$$1-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}-\frac{1}{7^{2n+1}}+\ \ldots\ =\frac{E_n\,\pi^{2n+1}}{2^{2n+2}(2n)!}.$$

where E_n is the nth Eulerian number, and the result corresponding to (4) is,

$$\frac{2^{2n+2}(2n)!}{E_n\,\pi^{2n+1}}-1=\frac{1}{3^{2n+1}}-\frac{(3^{2n+1}+1)}{3^{2n+1}.5^{2n+1}}$$

$$+\frac{(3^{2n+1}+1)(5^{2n+1}-1)}{3^{2n+1}.5^{2n+1}\,7^{2n+1}}+\frac{(3^{2n+1}+1)(5^{2n+1}-1)(7^{2n+1}+1)}{3^{2n+1}.5^{2n+1}.7^{2n+1}.11^{2n+1}}$$

$$-\frac{(3^{2n+1}+1)(5^{2n+1}-1)(7^{2n+1}+1)(11^{2n+1}+1)}{(3.5.7.11.13)^{2n+1}}-$$

$$++-+\qquad\ldots\quad(5)$$

The rule of signs enunciated above applies to (5) also.

4. Let σ_n denote the series*

$\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\ \ldots$ in which primes only occur, and S_n the series

$$\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\ \ldots=\sum_{r=1}^{\infty}\frac{1}{r^n}$$

Obviously
$$\sigma_n.S_n=\sum_{r=2}^{\infty}\frac{\lambda(r)}{r^n},$$

where $\lambda(r)$ denotes the number of prime factors of r. ... (6)

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* This series has been discussed by Glaisher, Q. J. M., Vol. XXV.

$$\therefore \qquad \sigma_n = \frac{1}{S_n} \cdot \sum_{r=2}^{\infty} \frac{\lambda(r)}{r^n}$$

$$= \sum_{r=1}^{\infty} \frac{\mu(r)}{r^n} \cdot \sum_{r=2}^{\infty} \frac{\lambda(r)}{r^n} \qquad \ldots \quad (7)$$

where $\mu(r) = 0$ if r has any repeated prime factor,

$= +1$ if r has an even number of dissimilar prime factors,

$= -1$ if r ,, odd ,, ,, ,,

[Bromwich: *Theory of Infinite Series*, p. 494., Ex. 45. (6)]

that is
$$\sum_{p=2}^{\infty} \frac{1}{p^n} = \sum_{r=1}^{\infty} \frac{\mu(r)}{r^n} \cdot \sum_{r=2}^{\infty} \frac{\lambda(r)}{r^n}. \qquad \ldots \quad (8)$$

Equating co-efficients of $\frac{1}{r^n}$ on both sides of this result, we get

$$\Sigma\ [\lambda(d_1)\ \mu(d_2)] = +1 \text{ or } 0^* \ \ldots \qquad \ldots \ (9)$$

according as r is or is not a prime, and the summation extends over all conjugate divisors of r (such that $d_1 \cdot d_2 = r$), and $\lambda(1)$ is taken to be zero.

Thus for $r = 48$,

$$\Sigma\ \lambda(d_1)\ \mu(d_2) =$$

$$\lambda(48)\ \mu(1) + \lambda(24)\ \mu(2) + \lambda(16)\ \mu(3) + \lambda(12)\ \mu(4)$$
$$+ \lambda(8)\ \mu(6) + \lambda(6)\ \mu(8) + \lambda(4)\ \mu(12) + \lambda(3)\ \mu(16)$$
$$+ \lambda(2)\ \mu(24)$$
$$= 2 - 2 - 1 + 0 + 1 + 0 + 0 + 0 = 0.$$

---

* 9) is obvious if r is a prime for then

$$\Sigma\ \lambda(d_1)\ \mu(d_2) = \lambda(r)\ \mu(1) = 1$$

If $r = m_k$ where m is a prime and k is a $+ve$ integer,

$$\Sigma\ \lambda(d_1)\ \mu(d_2) = \lambda(m^k)\ \mu(1) + \lambda(m^{k-1})\ \mu(m) + \lambda(m^{k-3})\ \mu(m^3)$$
$$+ \ldots + \lambda(m)\ \mu(m^{k-1}) = 1 - 1 + 0 + 0 + \ldots + 0 = 0.$$

5. Changing n into 2, 4, 6, 8, 10 ... in (7) and adding by columns, we get

$$\frac{1}{2^2-1}+\frac{1}{3^2-1}+\frac{1}{5^2-1}+\frac{1}{7^2-1}+\frac{1}{11^2-1}+\dots$$

$$= S(2)+S(4)+S(6)+S(8)+\quad\dots \tag{10}$$

where $$S(2r)=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{2r}}\cdot\sum_{n=2}^{\infty}\frac{\lambda(n)}{n^{2r}}.$$

Similarly,

$$\frac{1}{2^2+1}+\frac{1}{3^2+1}+\frac{1}{5^2+1}+\frac{1}{7^2+1}+\frac{1}{11^2+1}+\dots$$

$$= S(2)-S(4)+S(6)-S(8)+S(10)-+\dots\dots \tag{11}$$

From (10) and (11), we have

$$\frac{1}{2^4-1}+\frac{1}{3^4-1}+\frac{1}{5^4-1}+\frac{1}{7^4-1}+\frac{1}{11^4-1}+\frac{1}{13^4-1}+\dots$$

$$= S(4)+S(8)+S(12)+S(16)+\quad\dots \tag{12}$$

6. Glaisher has discussed the series on the left side of (10) in one of his papers in the *Quarterly Journal of Mathematics*, (Vols. 25, 27 and 28).

(*To be continued*).

S. D. S. CHOWLA

Solutions.

Questions 636 & 801.

(S. KRISHNASWAMI AIYANGAR, M.A.):—If

$$\sin e^x = \sum_0^\infty a_n \frac{x^n}{n!} \text{ and } \cos e^x = \sum_0^\infty b_n \frac{x^n}{n!},$$

show that $$a_n + a_{n+1} \log p + a_{n+2} \frac{(\log p)^2}{2!} + \ldots \ldots$$

$$= p - 3^n \frac{p^3}{3!} + 5^n \frac{p^5}{5!} \quad \ldots \ldots$$

and find $\sin(e^x + e^{-x})$.

(S. KRISHNASWAMI AIYANGAR, M.A.):—

If $$a_n = 1 - \frac{3^n}{3!} + \frac{5^n}{5!} - \frac{7^n}{7!} \quad \ldots \ldots$$

and $$-b_n = \frac{2^n}{2!} - \frac{4^n}{4!} + \frac{6^n}{6!} \quad \ldots \ldots \ldots$$

prove that

(1) $$a_n + a_{n+1} \log p + a_{n+2} \frac{(\log p)^2}{2!} \ldots \ldots$$

$$= p - 3^n \frac{p^3}{3!} + 5^n \frac{p^5}{5!} \ldots \ldots$$

(2) $\frac{1}{2} \sin(e^x + e^{-x})$

$$= \sum_0^\infty \left\{ a_{2n} b_0 + {}^{2n}C_2\, a_{2n-2} b_2 + \ldots + a_0 b_{2n} \right\} \frac{x^{2n}}{2n!}.$$

Solution by N. Sankara Aiyar.

Now $$\sin e^x = e^x - \frac{e^{3x}}{3!} + \frac{e^{5x}}{5!} \ldots \ldots \ldots$$

$$= \sum_0^\infty \frac{x^n}{n!} \left(1 - \frac{3^n}{3!} + \frac{5^n}{5!} \ldots\right)$$

showing that a_n is the same in Q. 636 and Q. 801.

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Also
$$\cos e^x = 1 - \frac{e^{2x}}{2!} + \frac{e^{4x}}{4!} \cdots \quad \cdots \quad \cdots$$

$$= 1 - \sum \frac{x^n}{n!} \left(\frac{2^n}{2!} - \frac{4^n}{4!} + \frac{6^n}{6!} \cdots \quad \cdots \right)$$

showing that b_n is also the same in both the questions, except when $n = 1$.

Now
$$a_n + a_{n+1} \log p + a_{n+2} \frac{(\log p)^2}{2!} \cdots \quad \cdots$$

$$= \sum_{0}^{r=\infty} \frac{(\log p)^r}{r!} \left(1 - \frac{3^{n+r}}{3!} + \frac{5^{n+r}}{5!} \cdots \cdots \right)$$

$$= \sum_{0}^{r=\infty} \frac{(\log p)^r}{r!} - \sum_{0}^{r=\infty} \frac{3^n}{3!} \frac{(3 \log p)^r}{r!} + \sum_{0}^{r=\infty} \frac{5^n}{5!} \frac{(5 \log p)^5}{r!} \cdots$$

$$= e^{\log p} - \frac{3^n}{3!} e^{3 \log p} + \frac{5^n}{5!} e^{5 \log p} \cdots \quad \cdots$$

$$= p - 3^n \cdot \frac{p^3}{3!} + 5^n \cdot \frac{p^5}{5!} \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

Again $\sin(e^x + e^{-x}) = \sin e^x \cos e^{-x} + \cos e^x \sin e^{-x}$

$$= \sum_{0}^{\infty} \frac{a_n}{n!} x^n \sum_{0}^{\infty} \frac{b_n}{n!} (-x)^n + \sum_{0}^{\infty} \frac{a_n}{n!} (-x)^n \sum_{0}^{\infty} \frac{b_n}{n!} x^n$$

$$= 2 \sum_{0}^{\infty} x^{2n} \left\{ \frac{a_{2n} b_0}{2n!} + \frac{a_{2n-2} b_2}{2n-2!\ 2!} + \frac{a_{2n-4} b_4}{2n-4!\ 4!} \cdots \cdots \right\}$$

$$= 2 \sum_{0}^{\infty} \frac{x^{2n}}{2n!} \left\{ a_{2n} b_0 + {}^{2n}C_2\, a_{2n-2} b_2 + \cdots \cdots \right\}$$

since all the odd powers cancel in the summation

Question 1267.

(RAJANARAYAN) :—Prove that if

$$e^n + \frac{1}{n} = e^m + \frac{1}{m}, \text{ then } e^{m+n} = e^m + e^n;$$

and conversely.

Remarks by A. Narasinga Rao.

There is some mistake in the statement of the question as the result is not true.

Take, for instance any particular value of m The equation

$$e^n + \frac{1}{n} = e^m + \frac{1}{m}$$

is transcendental in n and has an infinite number of roots n_1 n_2 n_3 ... one of which is, of course m. According to the question all these values of n satisfy the second relation

$$e^n = e^m / e^m - 1 = \text{a const. as } m \text{ is fixed.}$$

Hence $\qquad e^{n_1} = e^{n_2} = e^{n_3} = \dots \dots$

But $\qquad e^{n_1} + \frac{1}{n_1} = e^{n_2} + \frac{1}{n_2} = \dots \dots = e^m + \frac{1}{m}.$

$\therefore \qquad n_1 = n_2 = n_3 \dots \dots$ which is obviously untrue.

If the Question be taken to assert the existence of pairs of numbers m, n which satisfy the 2 relations

$$e^m + \frac{1}{m} = e^n + \frac{1}{n} \text{ and } e^m + e^n = e^{m+n}$$

then there is nothing remarkable in it, as we may treat the two equations as simultaneous in m and n and solve them.

Lastly let us see if the result is true if we restrict ourselves to real values. Take $e^m = 3$

so that

$$m = \log_e 3$$

$$e^n = \frac{3}{3-1} = 1.5$$

$$n = \log_e 1.5$$

Hence we should have

$$3 + \frac{1}{\log_e 3} = 1.5 + \frac{1}{\log_e 1.5}$$

or $\qquad 1.5 = \frac{1}{\log_e 1.5} - \frac{1}{\log_e 3} = \frac{\log_e 2}{(\log_e 3) \times (\log_e 1.5)}.$

The right side is found on calculation to exceed 1.55 and hence the equality cannot hold.

Question 1323.

(B. B. BAGI) :—Two circles intersect in A and B and a straight line through A cuts the circles in PQ. Prove that the circum-circle of the triangle PBQ envelops a cardioid.

Solution by M. V. Seshadri.

This can be easily proved by inversion. Inverting with respect to B and denoting the corresponding inverse points by accented letter we have the circles ABP, ABQ and PBQ inverting into straight lines $A'P'$, $A'Q'$ and $P'Q'$ and the line APQ into the circle $A'B'P'Q'$.

Now the lines $A'P'$ and $A'Q'$ are fixed in position and also the point B. By the property of Simson's lines the feet of the perpendiculars from B on $A'P'$, $A'Q'$ and $P'Q'$ are collinear, *i.e.* the foot of the perp. from B on $P'Q'$ lies on the line joining the feet of the perps. from B on $A'P'$ and $A'Q'$ (which is a fixed line). Hence the line $P'Q'$ envelops a parabola with B as focus. Now inverting this property we have in the original figure the circle PBQ enveloping a cardioid which is the inverse of a parabola with respect to its focus.

Question 1326.

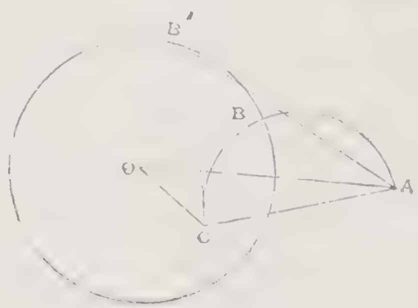
(N. DURAIRAJAN) :—If OA, OB, OC are three rods in a plane jointed at O, show how to determine the triangle ABC such that the radius of the circle ABC is a minimum.

Solution by the Proposer.

Let OA, OB, OC be the three rods hinged at O.

Let OA $>$ OB $>$ OC.

Now keep OA, OC fixed and move the rod OB, round O. The circum-radius R of the triangle ABC is $\dfrac{\frac{1}{2}AC}{\sin ABC}$.



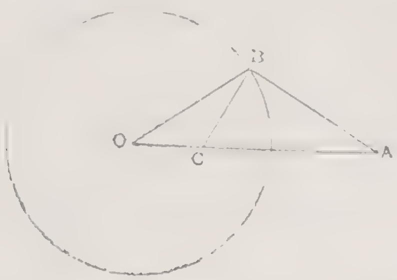
Hence R is a minimum when sin ABC is a maximum. The absolute maximum of sin ABC is unity, when ABC is 90°. In the present case, as A is outside and C is inside the circle centre O and radius OB, the circle on AC as diameter cuts this circle in two real points. Hence the least value of $R = \frac{1}{2}AC$.

Now AC has its least value when OA, OC lie are in the same direction, and in this case

$$R = \tfrac{1}{2}(OA - OC).$$

Hence the configuration giving the least value of R is such that OA, OC are on a line and on the same side of O and the triangle ABC is a right-angled triangle having B as a right-angle.

It is remarkable that this least value of R is independent of OB.



When A, B, say are kept fixed and C is moved round O, the circle on AB as diameter may not cut the circle OC in real points and the points at which AB subtends a maximum or minimum angle are those at which the two circles through A, B touch the circle (centre O and radius OC). Of these two points (C_1, C_2, say), that one will have to be selected, which is such that $\sin AC_1B > \sin AC_2B$. [As the reality of intersection cannot be easily determined, the case of OA, OC being fixed is here taken for simplifying the argument.]

Questions for Solution

1419. (S. D. CHOWLA) :—Prove that :

(i) $e^{z} = (1 + \tanh z)(1 + \tanh z_1)(1 + \tanh z_2) \ldots$ ad inf.,

where $e^{4z} - 2e^{2z}(2e^{2z_1} + 1) + 1 = 0$, and z_1, z_2 are connected by a similar relation, and so on.

(ii) $$\operatorname{cosec} \theta = \left(1 + \frac{1}{\sec^2 \theta + \tan^2 \theta}\right)$$

$$\left(1 + \frac{1}{\sec^2 \theta_1 + \tan^2 \theta_1}\right) \left(1 + \frac{1}{\sec^2 \theta_2 + \tan^2 \theta_2}\right) \ldots \text{ ad. inf.}$$

where $\tan \theta_1 = 2 \tan \theta \sec \theta$,

and in general $\tan \theta_n = 2 \tan \theta_{n-1} \sec \theta_{n-1}$.

1420. (S. D. CHOWLA) :—Evaluate the q series :

$$\frac{q^2}{1-q} \cdot \frac{1}{1+q^2} - \frac{q^6}{(1-q^2)(1-q^4)} \cdot \frac{1}{1+q^4} + \frac{q^{12}}{(1-q^2)(1-q^4)(1-q^6)} \cdot \frac{1}{1+q^6}$$

$$- \frac{q^{20}}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)} \cdot \frac{1}{1+q^8} + - + \ldots \text{ ad inf.}$$

where, as usual, $\log q = -\pi k'/k$.

In particular, find its value for $q = e^{-\pi\sqrt{1365}}$.

1421. (SELECTED) :—

A straight line PQRS intersects two confocal ellipses at the points P, Q, R, S in order. If the tangents at P and R are perpendicular, prove that the tangents at Q and S are also perpendicular.

1422. (A. NARASINGA RAO) :—Show that if three tangents to a tricusp hypocycloid are concurrent, then the three tangents perpendicular to these form a triangle whose pedal lines touch the tricusp.

1423. (V. RAMASWAMI AIYAR) :—ABCD is a spherical quadrilateral whose sides are supposed rigid and jointed at the corners and in which the sums of the opposite sides are equal. If one of the sides be held fixed and the quadrilateral deformed on the sphere, show that the locus of the *incentre* of the quadrilateral is a small circle whose pole is on that side, and whose radius is independent of which particular side is held fixed.

1424. (M. V. RAMAKRISHNAN):—Two circles intersect at P and Q A straight line APB cuts them again at A, B. Then the triangle AQB is obviously of given species.

Prove that

(1) any straight line connected invariably with the triangle AQB touches an ellipse unless it passes throuh Q ;

(2) and that any circle invariably connected with it envelops a limacon

1425. (M. V. RAMAKRISHNAN) :—If R, R_1, R_2, R_3 are the circum-radii of the four similar triangles whose sides touch a circle of radius r, show that

(1) $R_1 + R_2 + R_3 = R,$

(2) $R_1^{-1} + R_2^{-1} + R_3^{-1} - R^{-1} = 4r^{-1}.$

and (3) that the angles of the triangles are 2α, 2β, 2γ

where $$\tan\alpha = \left\{\frac{R_1^{-1} - R^{-1}}{R_2^{-1} + R_3^{-1}}\right\}^{\frac{1}{2}}, \text{ etc.}$$

THE INDIAN MATHEMATICAL SOCIETY.

Statement of Accounts for the Year 1925.

Receipts:—	Rs.	A.	P.	**Expenditure**:—	Rs.	A.	P.
Balance from 1924	4,362	6	4	Journal and other Printing ...	797	14	...
Subscription from Members ...	1,710	...	...	Books and Periodicals ...	549	12	5
„ from Subscribers ...	264	...	...	Library Expenses	400	4	...
Life Subscription	300	...	...	Ordinary Working Expenses ...	333	15	...
Interest on Investments ...	202	4	...	Closing Balance	4,860	14	11
Miscellaneous Receipts ...	104	2	...				
	6,942	12	4		6,942	12	4

MADRAS,
29th March 1926.

S. NARAYANA AIYAR,
Hon. Treasurer.

LIST OF JOURNALS RECEIVED

during January and February 1926.

1 American Mathematical Monthly, **32**, 10.*
2 Annales de la faculte des Sciences de L'universite de Toulouse 16, Annee 1924.
3 Annales Scientifiques de L'ecole Normale Superieure, Nov, 1925
4 Astrophysical Journal **62**, 4.
5 Bulletin of the American Math. Society, **31**, 9, 10 (2 copies).
6 Bulletin of the Calcutta Math. Soc. **16**, 2 (2 copies).
7 Bulletin des Sciences Math., Dec. 1925.
8 Encyclopaedie der Math. Wissenchaften Band **V**, 3, Heft 5.
9 Journal de Mathematiques, **4**, 3 and 4.
10 Journal fur die reine und angewandte Math, **155**, 4. 1.
11 Journal of the Science Association of Vizianagram, **2**, 2 and 3,
12 Mathematische Annalen, **95**, 3.
13 Mathematical Gazette, **12**, 179 (3 copies); **13**, 180 (2 copies).
14 Messenger of Mathematics, **55**, 4, 5 and 6 (2 copies).
15 Monthly Notices of the Royal Astronomical Society, **86**, 1. Also **75**, 2, 3, 4; **77**, 5; **82**, 1, 7 and 9; **78**, 1.
16 Nature, Nos. 2926 to 2931.
17 Philosophical Magazine, **1**, (7 Series); 1 (Jan. 1926).
18 Philosophical Transactions of the Royal Society of London, **225**, 632, 633 and 634.
19 Popular Astronomy, **33**, 10 and **34**, 1 (2 copies each).
20 Proceedings of the Cambridge Philosophical Society, **22**, 6.
21 Proceedings of the Royal Society, **109**, 752; **110**, 753.

Back numbers.

Journal de lecole Polytechnique

1st Series: 1—8; 11—38; 40—43; 45—64.

2nd Series: 1—24.

L'Intermediaire des Mathematiciens, 1—11; 13, 15, 12—25, 27.

A correction:—

In the October 1925 issue of the 19th Volume of Annales de l'observatoire de Paris Memoires though received in the Library was by oversight not acknowledged.

Books.

General Cyclides with special reference to quintic cyclide, by H. P. Pettitt.

Expansion Problems in connexion with Hyper Geometric differential equation, by B. P. Reinsch.

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* Numbers in black type refer to the Volume, and those in ordinary type to the specific number of the issue.

The Indian Mathematical Society

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